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CITATION:

Yokoyama, Keita. Notes on the first-order part of Ramsey's theorem for pairs (Proof theory and complexity). 数理解析研究所講究録 2013, 1832: 127-134

ISSUE DATE:

2013-04

URL:

<http://hdl.handle.net/2433/194849>

RIGHT:

# Notes on the first-order part of Ramsey's theorem for pairs

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## Abstract

We give the  $\Pi_2^0$ -part, the  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{RT}_2^2$  and related combinatorial principles.

## 1 Introduction

Determinating the first-order part of  $\text{WKL}_0 + \text{RT}_2^2$  and other important combinatorial principles is a one of the crucial topics in the study of Reverse Mathematics (see, e.g., [2, 4]). The usual approach for these questions is using forcing arguments to construct a second-order part for the target combinatorial principle. On the other hand, there is a traditional way to study the strength of combinatorial principles by using indicator functions. (For the details of indicator functions, see [6].) In [1], Bovykin and Weiermann gave the  $\Pi_2^0$ -part of  $\text{WKL}_0 + \text{RT}_2^2$  by means of an indicator function defined by a density notion, using the idea of Paris [7] and Paris/Kirby [8]. Using similar arguments, we can show that the  $\Pi_2^0$ -part of  $\text{WKL}_0^* + \text{RT}_2^2$  is equivalent to Elementary Function Arithmetic (see [9]). In this paper, we give the  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{RT}_2^2$  based on [1]. We will also give several density notions to characterize the  $\Pi_2^0$ -part, the  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{RT}_{<\infty}^2$ ,  $\text{SRT}_2^2$ ,  $\text{SRT}_{<\infty}^2$  and EM.

## 2 The $\Pi_2^0$ -part of $\text{WKL}_0 + \text{RT}_2^2$

This section is essentially due to Bovykin/Weiermann[1].

**Definition 2.1** (within  $\text{IS}_1$ ). For a finite set  $X$ , we define the notion of  $n$ -density as follows.

- A finite set  $X$  is said to be  $0$ -dense if  $|X| > \min X$ .
- A finite set  $X$  is said to be  $n + 1$ -dense if for any (coloring) function  $P : [X]^2 \rightarrow 2$ , there exists a subset  $Y \subseteq X$  such that  $Y$  is  $n$ -dense and  $Y$  is  $P$ -homogeneous, i.e.,  $P$  is constant on  $[Y]^2$ .

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Note that “ $X$  is  $m$ -dense” can be expressed by a  $\Sigma_0$ -formula.

**Definition 2.2.**  $n\text{PH}_2^2$  asserts that for any  $a$  there exists an  $n$ -dense set  $X$  such that  $\min X > a$ .

Define  $T_0 := \{k\text{PH}_2^2 \mid k \in \omega\} \cup \text{IS}_1$ .

**Lemma 2.1.** •  $\text{WKL}_0 + \text{RT}_2^2 \vdash n\text{PH}_2^2$  for any  $n \in \omega$ .

•  $\text{IS}_1 \vdash m\text{PH}_2^2 \rightarrow \text{PH}_{m+1}^2$ .

*Proof.* Easy. □

**Lemma 2.2** (Bovykin/Weiermann[1]). *Let  $M$  be a countable model of  $\text{IS}_1$ , and let  $X \subseteq M$  is a  $(M)$ -finite set which is  $k$ -dense for any  $k \in \omega$ . Then, there exists a cut  $I \subseteq M$  such that  $\min X \in I < \max X$ ,  $X \cap I$  is unbounded in  $I$  and  $(I, \text{Cod}(I/M) \models \text{WKL}_0 + \text{RT}_2^2$ .*

*Proof.* See [1]. □

**Theorem 2.3** (Bovykin/Weiermann[1]). *A  $\Pi_2^0$  sentence  $\psi$  is provable in  $\text{WKL}_0 + \text{RT}_2^2$  if and only if it is provable in  $T_0$ .*

*Proof.* See [1]. □

In fact, we can generalize this theorem as follows.

**Theorem 2.4.** *A  $\Pi_2^0$  formula  $\psi$  ( $\psi$  may contain set parameters) is provable in  $\text{WKL}_0 + \text{RT}_2^2$  if and only if it is provable in  $\text{IS}_1^0 \cup \{k\text{PH}_2^2 \mid k \in \omega\}$ . (Here,  $\text{IS}_1^0$  is a system of second-order arithmetic which contains basic axioms and induction axioms for  $\Sigma_1^0$ -formulas with set parameters.)*

### 3 The $\Pi_3^0$ -part of $\text{WKL}_0 + \text{RT}_2^2$

**Definition 3.1.** Let  $\theta(a, x, y)$  be a  $\Sigma_0$ -formula. We say that a finite set  $X = \{a_i \mid i \leq l\}$  dominates  $\theta(a, \cdot, \cdot)$  if  $\forall i < l \forall x \leq a_i \exists y \leq a_{i+1} \theta(a, x, y)$  holds. We define several variations of  $\text{PH}_2^2$  as follows:

- $\theta\text{-}n\text{PH}_2^2 := \forall a (\forall x \exists y \theta(a, x, y) \rightarrow \exists X (X \text{ is finite, } n\text{-dense, and dominates } \theta(a, \cdot, \cdot)))$ ,
- $\widetilde{n\text{PH}_2^2} := \forall X (\forall x \exists y \geq x \ y \in X \rightarrow \exists Y (Y \text{ is finite, } n\text{-dense, and } Y \subseteq X))$ .

Define  $T_1 := \{\theta\text{-}k\text{PH}_2^2 \mid k \in \omega, \theta \in \Sigma_0\} \cup \text{IS}_1$  and  $\widetilde{T}_1 := \{\widetilde{k\text{PH}_2^2} \mid k \in \omega\} \cup \text{RCA}_0$ . Note that  $T_1$  is a  $\Pi_3^0$ -theory, i.e.,  $T_1$  is a set of  $\Pi_3^0$ -sentences.

**Lemma 3.1.** *Let  $\theta(a, x, y)$  be a  $\Sigma_0$ -formula, and let  $n \in \omega$ . Then,  $\text{WKL}_0 + \text{RT}_2^2 \vdash \theta\text{-}n\text{PH}_2^2$ , and  $\text{WKL}_0 + \text{RT}_2^2 \vdash n\text{PH}_2^2$ .*

*Proof.* Easy. □

**Theorem 3.2.** A  $\Pi_3^0$  sentence  $\psi$  is provable in  $\text{WKL}_0 + \text{RT}_2^2$  if and only if it is provable in  $T_1$ . Thus,  $T_1$  is the  $\Pi_3^0$ -part of  $\text{WKL}_0 + \text{RT}_2^2$ .

*Proof.* We show that  $T_1 \not\vdash \psi$  implies  $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$  for any  $\Pi_3^0$ -sentence  $\psi$ . Assume that  $\psi \equiv \forall a \exists x \forall y \theta(a, x, y)$  is not provable from  $T_1$ . Then, there exists a nonstandard countable model  $M \models T_1$  such that  $M \models \forall x \exists y \neg \theta(a, x, y)$  for some  $a \in M$ . By  $(\neg \theta)$ - $k\text{PH}_2^2$  and overspill, there exists an  $m$ -dense set  $X$  which dominates  $\neg \theta(a, \cdot, \cdot)$  for some  $m \in M \setminus \omega$ . By Lemma 2.2, there exists an initial segment  $I \subseteq_e M$  such that  $(I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2$  and  $I \cap X$  is unbounded in  $I$ . Since  $X$  dominates  $\neg \theta$ , for any  $x \in I$  there exists  $y \in I$  such that  $I \models \neg \theta(a, x, y)$ . Thus, we have  $(I, \text{Cod}(I/M)) \models \neg \psi$ , which means that  $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$ .  $\square$

**Theorem 3.3.** A  $\Pi_3^0$  formula  $\psi$  is provable in  $\text{WKL}_0 + \text{RT}_2^2$  if and only if it is provable in  $\widetilde{T}_1$ .

*Proof.* Similar to the proof of Theorem 3.2.  $\square$

Note that  $\widetilde{T}_1$  is equivalent to  $\text{I}\Sigma_1^0 \cup \{\forall A \forall a (\forall x \exists y \theta(A, a, x, y) \rightarrow \exists X (X \text{ is finite, } n\text{-dense, and dominates } \theta(A, a, \cdot, \cdot))) \mid n \in \omega, \theta \in \Sigma_0^0\}$  with respect to  $\Pi_1^1$ -sentences.

## 4 The $\Pi_4^0$ -part of $\text{WKL}_0 + \text{RT}_2^2$

**Definition 4.1** (within  $\text{I}\Sigma_1$ ). Let  $\theta(a, x, y, z)$  be a  $\Sigma_0$ -formula. Then, we define the notion of *weakly domination* as follows.

- A 0-dense set  $X$  *weakly dominates*  $\theta(a, \cdot, \cdot, \cdot)$ .
- An  $n + 1$ -dense set  $X$  *weakly dominates*  $\theta(a, \cdot, \cdot, \cdot)$  if for any coloring  $P : [X]^2 \rightarrow 2$ , there exists a  $P$ -homogeneous set  $Y \subseteq X$  such that  $\forall x < \min X \exists y < \min Y \forall z < \max Y \theta(a, x, y, z)$ ,  $Y$  is  $n$ -dense and weakly dominates  $\theta(a, \cdot, \cdot, \cdot)$ .

Note that “ $X$  is  $m$ -dense and weakly dominates  $\theta(a, \cdot, \cdot, \cdot)$ ” can be expressed by a  $\Sigma_0$  formula.

**Definition 4.2.** Let  $\theta(a, x, y, z)$  be a  $\Sigma_0$ -formula. Then, the assertion  $\theta^* \text{-} n\text{PH}_2^2$  is the following

$$\forall a \forall b (\forall x \exists y \forall z \theta(a, x, y, z) \rightarrow \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(a, \cdot, \cdot, \cdot) \text{ and } \min X > b)).$$

Define  $T_2 := \{\theta^* \text{-} n\text{PH}_2^2 \mid n \in \omega, \theta(a, x, y, z) \in \Sigma_0\} \cup \text{I}\Sigma_1$ . Note that  $T_2$  is a  $\Pi_4^0$ -theory.

**Lemma 4.1.** Let  $\theta(a, x, y, z)$  be a  $\Sigma_0$ -formula, and let  $n \in \omega$ . Then,  $\text{WKL}_0 + \text{RT}_2^2 \vdash \theta^* \text{-} n\text{PH}_2^2$ .

*Proof.* Easy.  $\square$

**Theorem 4.2.** A  $\Pi_4^0$  sentence  $\psi$  is provable in  $\text{WKL}_0 + \text{RT}_2^2$  if and only if it is provable in  $T_2$ . Thus,  $T_2$  is the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{RT}_2^2$ .

*Proof.* We show that  $T_2 \not\vdash \psi$  implies  $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$  for any  $\Pi_4^0$ -sentence  $\psi$ . Assume that  $\psi \equiv \forall a \exists x \forall y \forall z \theta(a, x, y, z)$  is not provable from  $T_2$ . Then, there exists a nonstandard countable model  $M \models T_2$  such that  $M \models \forall x \exists y \neg \theta(a, x, y, z)$  for some  $a \in M$ . By  $(k, \neg\theta)\text{PH}_2^2$  and overspill, there exists an  $(m, \theta(a, \cdot, \cdot, \cdot))$ -dense set  $X$  such that  $\min X > a$  for some  $m \in M \setminus \omega$ . As the proof of Theorem 1 of [1], we can construct a descending sequence  $X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$  which satisfies the following:

- $I = \sup\{\min X_i \mid i \in \omega\} \subseteq_e M$ ,
- $(I, \text{Cod}(I/M)) \models \text{WKL}_0 + \text{RT}_2^2$ ,
- $I \cap X$  is unbounded in  $I$ ,
- $\forall x \leq \min X_i \exists y \leq \min X_{i+1} \forall z \leq \max X_{i+1} \neg \theta(a, x, y, z)$  for any  $i \in \omega$ .

Since  $\min X_i < \min X_{i+1} < I < \max X_{i+1}$  for any  $i \in \omega$ , we have  $I \models \forall x \exists y \forall z \neg \theta(a, x, y, z)$ , i.e.,  $(I, \text{Cod}(I/M)) \models \neg \psi$ . This means that  $\text{WKL}_0 + \text{RT}_2^2 \not\vdash \psi$ .  $\square$

**Remark 4.3.** Adding set parameters, we can easily show the following: a  $\Pi_4^0$  formula  $\psi$  is provable in  $\text{WKL}_0 + \text{RT}_2^2$  if and only if it is provable in

$$\text{IS}_1^0 \cup \{ \forall A \forall a \forall b (\forall x \exists y \forall z \theta(A, a, x, y, z) \rightarrow \exists X (X \text{ is } n\text{-dense, weakly dominates } \theta(A, a, \cdot, \cdot, \cdot) \text{ and } \min X > b) \mid n \in \omega, \theta \in \Sigma_0^0 \}.$$

## 5 $\text{PH}_2^2$ with stronger largeness notion

In this section, we compare  $n\text{PH}_2^2$  with  $\text{PH}_2^2$  plus “stronger largeness”.

**Definition 5.1** (within  $\text{IS}_1$ ). • A finite set  $X$  is said to be 0-large if  $X \neq \emptyset$ .

- A finite set  $X$  is said to be  $r + 1$ -large if there is a partition  $X = \bigsqcup_{i \leq \min X} Y_i$  such that  $\max Y_i < \min Y_{i+1}$  for any  $i < \min X$  and each  $Y_i$  is  $r$ -large.

**Remark 5.1.** 1. For any  $r \in \omega$ ,  $\text{IS}_1$  proves that for any  $a$ , there exists a finite set  $X$  such that  $\min X > a$  and  $X$  is  $r$ -large.

2.  $Q(a, b) := \max\{r \mid [a, b] \text{ is } r\text{-large}\}$  is an indicator function for  $\text{WKL}_0$ .

3. More generally, if  $M$  is a model of  $\text{IS}_1$  and  $X \subseteq M$  is  $r$ -large for some  $r \in M \setminus \omega$ , then there exists a cut  $I \subseteq_e M$  such that  $(I, \text{Cod}(I/M)) \models \text{WKL}_0$  and  $X \cap I$  is unbounded in  $I$ .

**Definition 5.2.** 1.  $\text{PH}_{2,r}^2$  asserts that for any  $a$ , there exists a finite set  $X$  such that  $\min X > a$  and for any coloring  $P : [X]^2 \rightarrow 2$ , there exists a  $P$ -homogeneous set  $Y \subseteq X$  which is  $r$ -large.

2.  $\widetilde{\text{PH}}_{2,r}^2$  asserts that for any infinite set  $A$ , there exists a finite set  $X$  such that  $X \subseteq A$  and for any coloring  $P : [X]^2 \rightarrow 2$ , there exists a  $P$ -homogeneous set  $Y \subseteq X$  which is  $r$ -large.
3. In general,  $\widetilde{n\text{PH}}_{2,r}^2$  asserts that for any infinite set  $A$ , there exists a finite set  $X$  such that  $X \subseteq A$  and  $X$  is  $(n, r)$ -dense, where the notion of  $(n, r)$ -density is defined as follows:
  - A finite set  $X$  is said to be  $(0, r)$ -dense if  $X$  is  $r$ -large.
  - A finite set  $X$  is said to be  $(n+1, r)$ -dense if for any coloring  $P : [X]^2 \rightarrow 2$ , there exists a  $P$ -homogeneous set  $Y \subseteq X$  which is  $(n, r)$ -dense.

**Proposition 5.2.**  $\text{IS}_1 \vdash n\text{PH}_2^2 \rightarrow \text{PH}_{2,n}^2$ .

*Proof.* Easy. □

The strength of  $\text{PH}_{2,r}^2$  is related to the strength of  $n\text{PH}_2^2$  in the following meaning.

**Proposition 5.3.** Assume that  $\text{WKL}_0 \vdash \widetilde{\text{PH}}_{2,r}^2$  for all  $r \in \omega$ , then we have  $\text{WKL}_0 \vdash \widetilde{n\text{PH}}_2^2$  for all  $n \in \omega$ .

*Proof.* Our assumption is  $\text{WKL}_0 \vdash \widetilde{1\text{PH}}_{2,r}^2$  for any  $r \in \omega$ . We will show by induction on  $n$  that  $\text{WKL}_0 \vdash \widetilde{n\text{PH}}_{2,r}^2$  for any  $r \in \omega$  and for any  $n \in \omega$ . Let  $\text{WKL}_0 \vdash \widetilde{n\text{PH}}_{2,r}^2$  for any  $r \in \omega$ . Assume for the sake of contradiction that  $\text{WKL}_0 \not\vdash \widetilde{(n+1)\text{PH}}_{2,r}^2$  for some  $r \in \omega$ . Then, there exists a model  $(M, S) \models \text{WKL}_0$  and  $A \in S$  such that  $M \not\cong \omega$ ,  $A$  is unbounded in  $M$  and any  $(M)$ -finite subset of  $A$  is not  $(n+1, r)$ -dense. By the assumption, there exists an  $(n, s)$ -dense subset of  $A$  for any  $s \in \omega$ . Thus, by overspill, for some  $m \in M \setminus \omega$ , we can take an  $(n, m)$ -dense subset  $X \subseteq A$ . We will show that this  $X$  is in fact  $(n+1, r)$ -dense, which leads to a contradiction. By the definition of  $(n, m)$ -density, for any coloring  $P : [X]^2 \rightarrow 2$ , there exists a  $P$ -homogeneous set  $Y_1 \subseteq X$  which is  $(n-1, m)$ -dense, and we can repeat this process  $n$ -times then the result set  $Y_n$  is  $m$ -large. By Remark 5.1.3, there exists a cut  $I \subseteq_e M$  such that  $(I, \text{Cod}(I/M)) \models \text{WKL}_0$  and  $Y_n \cap I$  is unbounded in  $I$ . Thus, there exists a finite subset of  $Y_n \cap I$  which is  $(1, r)$ -dense. This means that  $Y_n$  is  $(1, r)$ -dense, and hence  $X$  is  $(n+1, r)$ -dense. □

Thus, if  $\text{WKL}_0 \vdash \widetilde{\text{PH}}_{2,r}^2$ , then  $\text{WKL}_0 + \text{RT}_2^2$  is a  $\Pi_2^0$ -conservative extension of  $\text{WKL}_0$ . This may give a new approach to study the proof-theoretic strength of  $\text{WKL}_0 + \text{RT}_2^2$ .

**Question 5.3.** Is  $\text{IS}_1 \cup \{n\text{PH}_2^2 \mid n \in \omega\}$  equivalent to  $\text{IS}_1 \cup \{\text{PH}_{2,r}^2 \mid r \in \omega\}$ ?

## 6 Other combinatorial principles

In this section, we give several density notions for  $\text{SRT}_2^2$ ,  $\text{RT}_{<\infty}^2$ ,  $\text{SRT}_{<\infty}^2$ , EM and ADS. (For the definitions of these combinatorial principles, see [2, 5, 1].) Using these notions, we can characterize  $\Pi_2^0$ ,  $\Pi_3^0$  or  $\Pi_4^0$  part of the target combinatorial principle as in Sections 2,3 and 4.

We reason within  $\text{IS}_1$ .

**Proposition 6.1.** *The  $\Pi_2^0$ -part,  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{SRT}_2^2$  is characterized by the following density notion.*

*A finite set  $X$  is said to be*

- 0-dense if  $|X| > \min X$ , and
- $m+1$ -dense if for any  $P : [X]^2 \rightarrow 2$ ,
  - there exists a  $P$ -homogeneous subset  $Y \subseteq X$  which is  $m$ -dense, or,
  - there exists  $Y = \{y_0 < y_1 < \dots < y_l\} \subseteq X$  such that  $P(y_0, y_i) \neq P(y_0, y_{i+1})$  for any  $0 < i < l$  and  $Y$  is  $m$ -dense.

For the strength of  $\text{SRT}_2^2$ , see also Chong/Slaman/Yang [3].

**Proposition 6.2.** *The  $\Pi_2^0$ -part,  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{RT}_{<\infty}^2$  is characterized by the following density notion.*

*A finite set  $X$  is said to be*

- 0-dense if  $|X| > \min X$ , and
- $m+1$ -dense if for any coloring  $P : [X]^2 \rightarrow k$  such that  $k < \min X$ , there exists a  $P$ -homogeneous subset  $Y \subseteq X$  which is  $m$ -dense.

**Proposition 6.3.** *The  $\Pi_2^0$ -part,  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{SRT}_{<\infty}^2$  is characterized by the following density notion.*

*A finite set  $X$  is said to be*

- 0-dense if  $|X| > \min X$ , and
- $m+1$ -dense if for any coloring  $P : [X]^2 \rightarrow k$  such that  $k < \min X$ ,
  - there exists a  $P$ -homogeneous subset  $Y \subseteq X$  which is  $m$ -dense, or,
  - there exists  $Y = \{y_0 < y_1 < \dots < y_l\} \subseteq X$  such that  $P(y_0, y_i) \neq P(y_0, y_{i+1})$  for any  $0 < i < l$  and  $Y$  is  $m$ -dense,

**Proposition 6.4.** *The  $\Pi_2^0$ -part,  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{EM}$  is characterized by the following density notion.*

*A finite set  $X$  is said to be*

- 0-dense if  $|X| > \min X$ , and
- $m+1$ -dense if
  - for any coloring  $P : [X]^2 \rightarrow 2$ , there exists  $Y \subseteq X$  such that  $P$  is transitive on  $Y$  and  $Y$  is  $m$ -dense, and,
  - there is a partition  $X = \bigsqcup_{i \leq \min X} Y_i$  such that  $\max Y_i < \min Y_{i+1}$  for any  $i < \min X$  and each  $Y_i$  is  $m$ -dense.

Here, a coloring  $P$  is said to be transitive if  $P(a, b) = P(b, c) \Rightarrow P(a, b) = P(a, c)$ .

**Proposition 6.5.** *The  $\Pi_2^0$ -part,  $\Pi_3^0$ -part and the  $\Pi_4^0$ -part of  $\text{WKL}_0 + \text{ADS}$  is characterized by the following density notion.*

*A finite set  $X$  is said to be*

- 0-dense if  $|X| > \min X$ , and
- $m+1$ -dense if for any transitive coloring  $P : [X]^2 \rightarrow 2$ , there exists a  $P$ -homogeneous subset  $Y \subseteq X$  which is  $m$ -dense.

In fact, Slaman/Chong/Yang[4] showed that  $\text{WKL}_0 + \text{ADS}$  is a  $\Pi_1^1$ -conservative extension of  $\text{B}\Sigma_2^0$ . Thus, for any  $n \in \omega$ ,  $\text{WKL}_0$  actually proves for any  $a$ , there exists a finite set  $X$  such that  $\min X > a$  and  $X$  is  $n$ -dense for  $\text{ADS}$ .

## Acknowledgments

The author would like to thank Dr. Tin Lok Wong for useful comments. This work was partially supported by a Japan Society for the Promotion of Science postdoctoral fellowship for young scientists, and by a grant from the John Templeton Foundation.

## References

- [1] Andrey Bovykin and Andreas Weiermann. The strength of infinitary ramseyan principles can be accessed by their densities. to appear.
- [2] Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66(1):1–55, 2001.
- [3] C. T. Chong, Theodore A. Slaman, and Yue Yang. The metamathematics of Stable Ramsey's Theorem for pairs. to appear.
- [4] C. T. Chong, Theodore A. Slaman, and Yue Yang.  $\Pi_1^1$ -conservation of combinatorial principles weaker than Ramsey's theorem for pairs. *Advances in Mathematics*, 230(3):1060–1077, 2012.



- [5] Denis R. Hirschfeldt and Richard A. Shore. Combinatorial principles weaker than Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 72(1):171–206, 2007.
- [6] Richard Kaye. *Models of Peano Arithmetic*. Oxford Logic Guides, 15. Oxford University Press, 1991. x+292 pages.
- [7] J. B. Paris. Some independence results for Peano Arithmetic. *Journal of Symbolic Logic*, 43(4):725–731, 1978.
- [8] J. B. Paris and L. A. S. Kirby.  $\Sigma_n$ -collection schemas in arithmetic. In *Logic Colloquium '77 (Proc. Conf., Wroclaw, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, 1978.
- [9] Keita Yokoyama. On the strength of Ramsey's theorem without  $\Sigma_1$ -induction. to appear.